Pergamon

Int. J. Solids Structures Vol. 35, No. 33, pp. 4477–4489, 1998

0020–7683/98 $19.00 + .00

PII: S0020–7683(97)00255–2

ANALYTICAL DERIVATION AND INVESTIGATION OF THE INTERFACE CRACK MODELS

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(*Received* 15 *March* 1997; *in revised form* 4 *August* 1997)

Abstract—The oscillating and the contact zone models are derived from the exact solution of the plane problem for an interface crack with contact zone of arbitrary length and procedure of transition from one model to another is demonstrated. The transcendental equations and approximate relations for the contact zone length determination in terms of the stress intensity factors of the oscillating model are found and the examples of these relations utilization are presented. The energy release rate (ERR) for both models are given in simple analytical form and slight difference between them for any contact zone length is found out. Quasi-invariance of the ERR with respect to contact zone length for any load and material properties is proved and a simple way of the ERR numerical determination is suggested. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

An interface crack investigation is primarily important for the strength of the composite assessment because interfacial and integranular fracture is common for such materials and usually define the material overall strength qualities. One of the two idealizations of the crack face boundary conditions at its tip is mainly used for an interface crack analysis. The first idealization developed in the papers of Williams (1959), Cherepanov (1962), Erdogan (1963), England (1965), Rice and Sih (1965) is based on the traction-free crack faces assumption and leads to the oscillating singularity at the crack tips causing contact and mathematical inpenetration of the near-tip crack faces. The second approach initiated in the numerical manner by Comninou (1977, 1978), Comninou and Schmueser (1979), assumes a closed form of the crack near its tips and produces crack faces contact length essentially dependent on the intensity of the external shear field. This model was studied analytically in the papers of Atkinson (1982), Simonow (1985, 1990), Gautesen and Dundurs (1987, 1988a, b), Loboda (1993), Gautesen (1995) and with the account of friction in the contact area it was studied by Antipov (1995). Beside the mentioned approaches the model of thin elastic interface with crack was suggested by Atkinson (1977) and developed by Delale and Erdogan (1988). An overview of the main results of the interface crack problem investigation has been done by Comninou (1990).

In spite of a great number of essential results on the considered problem there are still some difficulties in the interface crack fracture parameters determination for a finite size body. One is related to the cases of an essential shear field causing a long contact zone when the oscillating model can be doubtfully applied. But the contact model applicability is complicated by the necessity of the contact zone length determination which is rather difficult for a finite size body. In a case of a small contact zone the way of the contact model utilization free from the contact zone length determination was suggested by Loboda (1993). An application of this way for a cylinder containing a penny-shaped crack in the fixed end area was demonstrated by Loboda and Sheveleva (1995).

In this paper above-mentioned new approach based upon the quasi-invariance of the ERR with respect to the contact zone length is generalized for an arbitrary contact zone length. Besides the simple formula for ERR is found and the way of the approximate determination of the contact zone length via the stress intensity factors (SIFs) of the oscillating model is suggested.

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2. CONTACT ZONE AND OSCILLATING SOLUTIONS FOR A CRACK BETWEEN TWO DISSIMILAR HALF-PLANES

An interface crack between dissimilar materials as shown in Fig. 1 is considered. We assume σ, $\sigma_{x_1}^{\infty}$, $\sigma_{x_2}^{\infty}$ satisfy the continuity conditions defined by Rice and Sih (1965), and the crack surfaces are traction free for $x \in [c, a] = L_1$, and they are in the frictionless contact for $x \in (a, b) = L_2$. Position of the point a is arbitrary for a time. Without loss of generality, we can take

$$\gamma = \frac{\mu_1 + \mu_2 \kappa_1}{\mu_2 + \mu_1 \kappa_2} > 1$$

and $\tau \leqslant 0$ or $\gamma \leqslant 1$ and $\tau > 0$. In this case the longer contact zone arises at the right crack tip and that is why we take only this zone into account. It was shown by Gautesen and Dundurs (1988b) that oscillating singularity at the left crack tip does not essentially influence to the stress–strain field at the right crack tip.

By using the method described by Loboda (1993) the following formulas for the stresses and displacement derivatives needed in subsequent analysis have been found

$$x \in L_2: \quad \sigma_y = \frac{P(x)}{\sqrt{(x-c)(b-x)}}\left[\frac{1-\gamma}{1+\gamma}\cosh\phi_0(x) + \sinh\phi_0(x)\right] + \frac{Q(x)}{\sqrt{(x-c)(x-a)}}\left[\cosh\phi_0(x) + \frac{1-\gamma}{1+\gamma}\sinh\phi_0(x)\right], \quad (1)$$

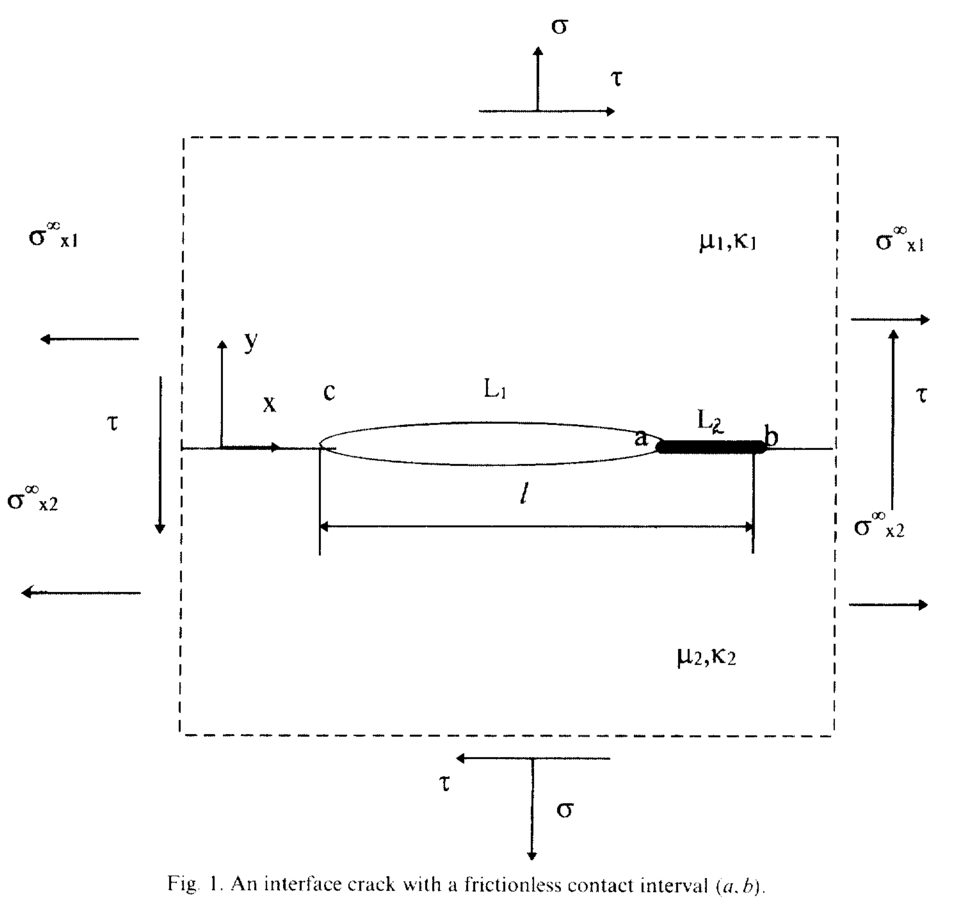

Fig. 1. An interface crack with a frictionless contact interval (a, b).

$$x > b: \quad \sigma_y - i\tau_{xy} = \frac{Q(x)\cos\phi(x)}{\sqrt{(x-c)(x-a)}} - \frac{P(x)\sin\phi(x)}{\sqrt{(x-c)(x-b)}} + i\left[\frac{P(x)\cos\phi(x)}{\sqrt{(x-c)(x-b)}} + \frac{Q(x)\sin\phi(x)}{\sqrt{(x-c)(x-a)}}\right], \tag{2}$$

$$x \in L_1: \quad 2\mu_1[v'_x] = \frac{\mu_1 + \kappa_1\mu_2}{\mu_2\sqrt{\gamma}}\left[\frac{P(x)\sin\phi^*(x)}{\sqrt{(x-c)(b-x)}} - \frac{Q(x)\cos\phi^*(x)}{\sqrt{(x-c)(a-x)}}\right], \tag{3}$$

where

$$\phi(x) = 2e\ln\frac{\sqrt{\lambda(x-c)}}{\sqrt{(x-a)}+\sqrt{(1-\lambda)(x-b)}},$$

$$\phi_0(x) = 2e\tan^{-1}\sqrt{\frac{\lambda(b-x)}{(x-a)}},$$

$$\phi^*(x) = 2e\ln\frac{\sqrt{\lambda(x-c)}}{\sqrt{(a-x)}+\sqrt{(1-\lambda)(b-x)}},$$

$$e = \frac{1}{2\pi}\ln\gamma, \quad [f] = f^+ - f^-,$$

parameter $\lambda = (b-a)/(b-c)$ describes the relative length of the contact zone at the right crack tip.

Polynomials $P(x)$ and $Q(x)$ were found from the conditions at the infinity and can be written in the form

$$P(x) = C_1x + C_2, \quad Q(x) = D_1x + D_2,$$

where

$$D_1 = \sigma\cos\beta - \tau\sin\beta, \quad C_1 = -\tau\cos\beta - \sigma\sin\beta,$$

$$D_2 = \beta_1 C_1 - \frac{c+a}{2}D_1, \quad C_2 = -\frac{c+b}{2} - \beta_1 D_1,$$

$$\beta = e\ln\frac{1-\sqrt{1-\lambda}}{1+\sqrt{1-\lambda}}, \quad \beta_1 = e\sqrt{(a-c)(b-c)}.$$

The stress intensity factors

$$k_1 = \lim_{x\to a+0}\sqrt{2(x-a)}\sigma_y(x,0), \quad k_2 = \lim_{x\to b+0}\sqrt{2(x-b)}\tau_{xy}(x,0),$$

due to the last formulae have the following form

$$k_1 = \frac{\sqrt{2\gamma l}}{\gamma+1}\sigma[\sqrt{1-\lambda}(\cos\beta - \delta\sin\beta) - 2e(\delta\cos\beta + \sin\beta)],$$

$$k_2 = \sigma\sqrt{\frac{l}{2}}[\delta\cos\beta + \sin\beta + 2e\sqrt{1-\lambda}(\cos\beta - \delta\sin\beta)], \tag{4}$$

where $\delta = \tau/\sigma$ and $l = b-c$ is the crack length.

The obtained solution is valid for any values of parameter a from $[c, b]$. But this solution is physically correct if the following additional conditions are satisfied

$$\sigma_y(x, 0) \leqslant 0 \quad \text{for } x \in L_2 \quad \text{and} \quad [v(x, 0)] \geqslant 0 \quad \text{for } x \in L_1, \tag{5}$$

(excluding zone of oscillation near the left crack tip). To satisfy the last inequalities we take $k_1 = 0$ that leads to the following equation

$$\beta = -\tan^{-1}\frac{2e}{\sqrt{1-\lambda}} - \chi + \pi(m-0.5), \quad \chi = \tan^{-1}\delta \tag{6}$$

with respect to λ. Analysis showed that for the relations (5) validity we should take $m = 0$ for $e > 0$ and $m = 1$ for $e < 0$. In this case the maximum root of eqn (6) in (0, 1) will be found. More over for a small root of (6) due to assumptions $1+\sqrt{1-\lambda} \approx 2$, $\tan^{-1}(2e/\sqrt{1-\lambda}) \approx \tan^{-1}(2e)$ the following asymptotic formula can be applied

$$\bar{\lambda}_0 = 4\exp\left[-\frac{1}{e}(\tan^{-1}(2e) - \chi + \pi(m-0.5))\right].$$

Particularly for $e > 0$ ($m = 0$) the last formula is reduced to

$$\bar{\lambda}_0 = 4\exp[[-\tan^{-1}(2e) - (\chi + \pi/2)]/e]. \tag{7}$$

The roots λ_0 of (6) were obtained numerically and their asymptotic values $\bar{\lambda}_0$ for $\gamma = 3$ and various δ are given in Table 1 and for $\gamma = 1.8$ they are given in Table 2. It is clear from Tables 1 and 2 that with good accuracy the asymptotic formula (7) can be used for λ_0 determination in the range of $\lambda_0 \leqslant 0.01$. It should be noted as well that the results of λ_0 determination are in good agreement with the correspondent results by Gautesen and Dundurs (1988a) (for example for $\gamma = 3$ and $\delta \to -\infty$ their value rounded to three digits is 0.329).

Table 1. Exact λ_0 and asymptotic $\bar{\lambda}_0$ values of the contact zone lengths for $\gamma = 3$ and various shear fields

δ	λ_0	$\bar{\lambda}_0$
0	$0.7327 \cdot 10^{-4}$	$0.7327 \cdot 10^{-4}$
-1	$0.6503 \cdot 10^{-2}$	$0.6542 \cdot 10^{-2}$
-2.5	$0.6072 \cdot 10^{-1}$	$0.6628 \cdot 10^{-1}$
-5	0.1502	0.1889
-10	0.2285	0.3303
-100	0.3182	0.5516
$\to -\infty$	0.3291	0.5841

Table 2. Exact λ_0 and asymptotic $\bar{\lambda}_0$ values of the contact zone lengths for $\gamma = 1.8$ and various shear fields

δ	λ_0	$\bar{\lambda}_0$
0	$0.2825 \cdot 10^{-7}$	$0.2825 \cdot 10^{-7}$
-1	$0.1252 \cdot 10^{-3}$	$0.1251 \cdot 10^{-3}$
-2.5	$0.9256 \cdot 10^{-2}$	$0.9482 \cdot 10^{-2}$
-5	$0.6121 \cdot 10^{-1}$	$0.6714 \cdot 10^{-1}$
-10	0.1498	0.1908
-100	0.2937	0.4977
$\to -\infty$	0.3123	0.5539

It is important that the solutions (1)–(3) remain applicable for any contact zone length and particularly for $a = b$. Namely for $a = b$ and $x > b$ the expression (2) for $\sigma_y - i\tau_{xy}$ can be written in the form

$$\sigma_y - i\tau_{xy} = \frac{1}{\sqrt{(x-c)(x-b)}}[Q(x)\cos\phi(x) - P(x)\sin\phi(x) + i(P(x)\cos\phi(x) + Q(x)\sin\phi(x))],$$

where

$$Q(x) = \left(x - \frac{c+b}{2}\right)(\sigma\cos\beta - \tau\sin\beta) - \beta_1(\sigma\sin\beta + \tau\cos\beta),$$

$$P(x) = -\left(x - \frac{c+b}{2}\right)(\sigma\sin\beta + \tau\cos\beta) - \beta_1(\sigma\cos\beta - \tau\sin\beta).$$

Taking into account that

$$\beta - \phi(x) = 2e\ln\frac{\sqrt{x-a} + \sqrt{(1-\lambda)(x-b)}}{\sqrt{x-c}(1+\sqrt{1-\lambda})}$$

for $a = b$ we arrive $\beta - \phi(x) = e\ln[(x-b)/(x-c)]$ and

$$\sigma_y - i\tau_{xy} = \frac{1}{\sqrt{(x-c)(x-b)}}\left(\left[\left(x - \frac{c+b}{2}\right)(\sigma\cos\omega - \tau\sin\omega) - el(\sigma\sin\omega + \tau\cos\omega)\right] - i\left[\left(x - \frac{c+b}{2}\right)(\sigma\sin\omega + \tau\cos\omega) + el(\sigma\cos\omega - \tau\sin\omega)\right]\right), \quad (8)$$

where $\omega = e\ln[(x-b)/(x-c)]$.

In the similar way formula (3) for $a = b$ can be reduced to the form

$$2\mu_1[v_x'] = -\frac{\mu_1 + \kappa_1\mu_2}{\mu_2\sqrt{\gamma}}\frac{1}{\sqrt{(x-c)(b-x)}}\left[\left(x - \frac{c+b}{2}\right)(\sigma\cos\omega^* - \tau\sin\omega^*) - el(\sigma\sin\omega^* + \tau\cos\omega^*)\right], \quad (9)$$

where $\omega^* = e\ln[(b-x)/(x-c)]$. The formulas (8) and (9) present the well-known oscillating solution for an interface crack. Due to applicability of the solution (1)–(3) both for the contact ($\lambda = \lambda_0$) and the oscillating ($\lambda = 0$) models we will use this solution now for the demonstration of the process of these models derivations.

In Fig. 2 the values of $\sigma_y(x, 0)/\sigma$ for $x \in L_2$, $\kappa = 2.8$, $\tau = 0$, $b = -c = 1$ and the various values of λ are shown. It is clear from these results that for $2\lambda = 10^{-3}$ longer part of the contact zone ($\approx 0.7(b-a)$) is in tension. Decreasing of λ leads to the relative length of the compressed zone increasing (for $2\lambda = 10^{-4}$ approximately $0.9(b-a)$ is compressed). For $\lambda \leqslant \lambda_0$ first inequality of (5) is valid but only for $\lambda = \lambda_0$ we have $\sigma_y(a, 0) = 0$.

Figures 3 and 4 illustrate the procedure of the second inequality of (5) satisfaction. Particularly in Fig. 3 the diagrams of

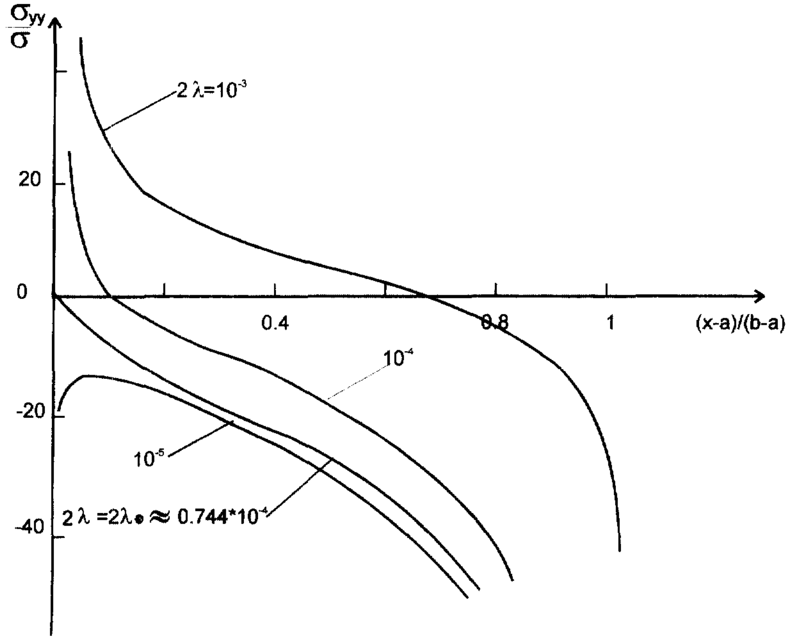

Fig. 2. Normal stress in the contact zone for various lengths.

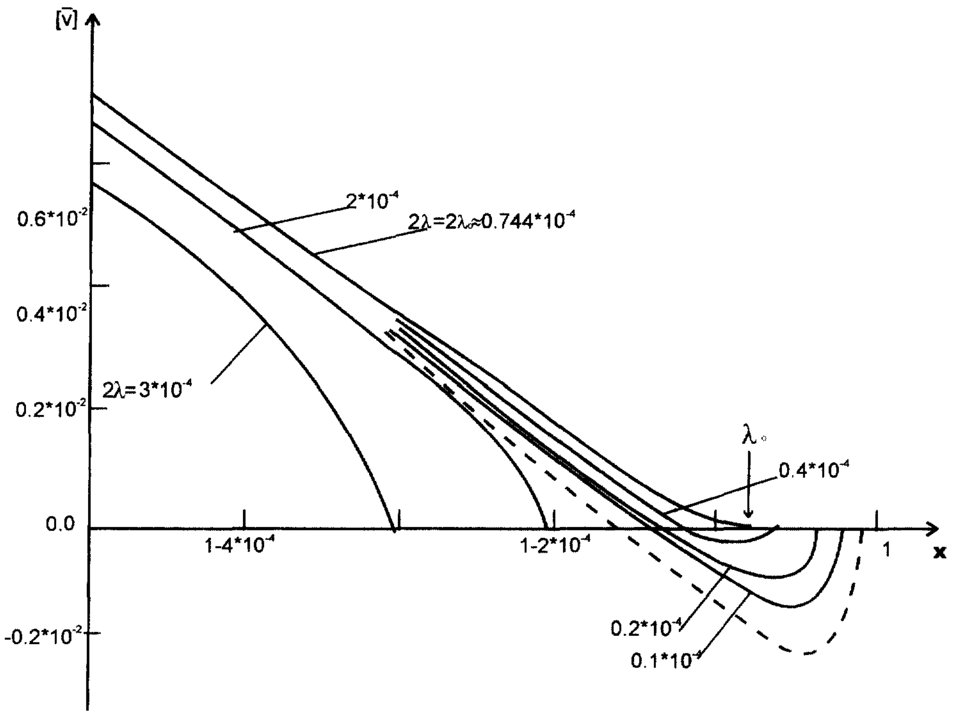

Fig. 3. Displacement jump at the right crack tip for a pure tension field and various contact zone lengths.

$$[\bar{v}(x)] = [v(x,0)] \bigg/ \left(\sigma \frac{\mu_1 + \mu_2 \kappa_1}{2\mu_1 \mu_2 \sqrt{\gamma}}\right)$$

in the left neighborhood of the right crack tip are given for various λ and $\kappa = 2.8$, $\tau = 0$,

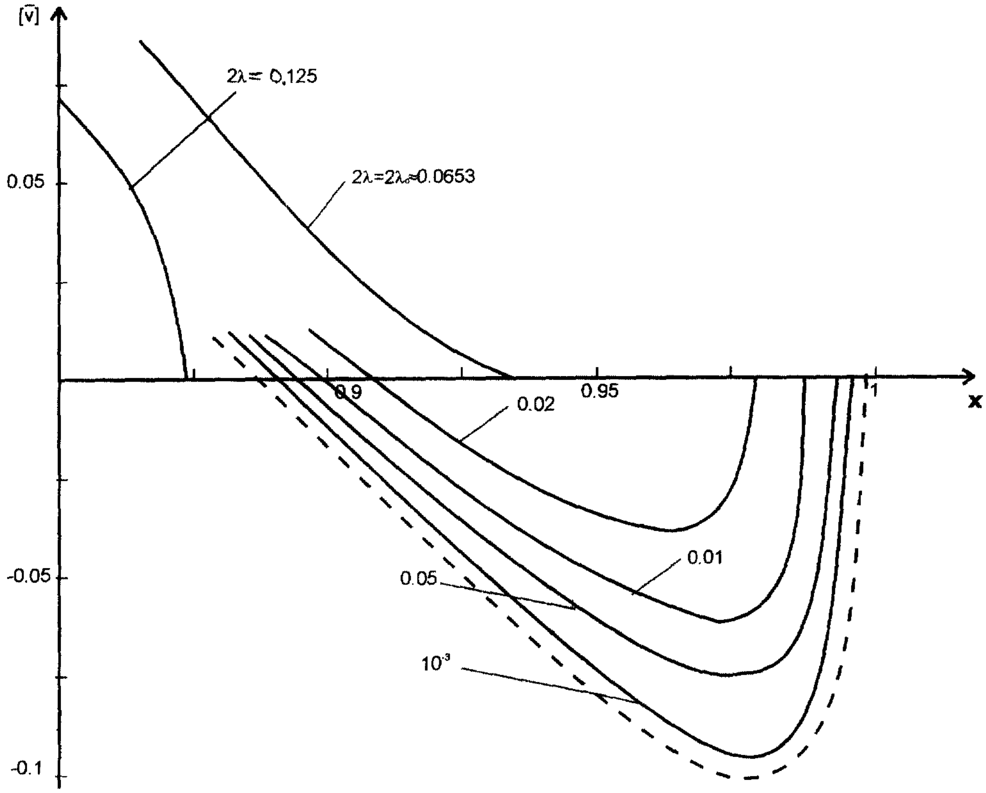

Fig. 4. Displacement jump at the right crack tip for a tension–shear field and various contact zone lengths.

$b = -c = 1$. One can clearly see that for $\lambda \geqslant \lambda_0$ (for $2\lambda = 3\cdot 10^{-4}$, $2\cdot 10^{-4}$, 10^{-4}) the part L_1 of the crack is opened and the second inequality of (5) is valid. Particularly for $\lambda > \lambda_0$ $[\bar{v}'(x)] \to \infty$ when $x \to a-0$, but for $\lambda = \lambda_0$ the jump $[\bar{v}'(a)] = 0$. Next for $\lambda < \lambda_0$ overlapping of crack tips appears and decreasing of λ leads to the increasing of the overlapping zone length and its amplitude. Finally, the dashed line corresponds to the classical (oscillating) solution (9). Similar effects can be seen from Fig. 4 where diagrams of

$$[\bar{v}(x)] = [v(x,0)] \Bigg/ \left[\sqrt{\sigma^2+\tau^2}\,\frac{\mu_1+\mu_2\kappa_1}{2\mu_1\mu_2\sqrt{\gamma}}\right]$$

for various λ, $\kappa = 2.8$, $\delta = -2$, and $b = -c = 1$ are displayed. In this case $\lambda_0 = 0.0351$, i.e. contact macro zone arises. Crack faces overlapping amplitude in this case is more essential than in Fig. 3 and it is not negligible small with respect to the max $[v(x,0)]$ in $[c,a]$. It can be explicitly seen from Figs 2–4 that the both inequalities (5) are satisfied only for $\lambda = \lambda_0$.

It is interesting to note that crack faces overlapping zones are larger than the real contact zone length. This difference is the largest for the pure oscillating solutions (dashed lines). In this case the analytical relationship between the asymptotic contact zone length $\bar{\lambda}_0 l$ and the zone of crack faces overlapping length

$$r_c = l\exp\left[-(\chi+\pi/2)/e\right] \tag{10}$$

obtained by Rice (1988) is the following

$$\bar{\lambda}_0 l = 4\exp\left[-\tan^{-1}(2e)/e\right]r_c \approx 4\exp(-2)r_c \approx 0.5413r_c. \tag{11}$$

But for a large contact zone length according to Table 2 and Fig. 4 their difference is more essential, because λ_0 is usually less than $\bar{\lambda}_0$. Qualitively the same conclusion concerning the relationship between the contact zone lengths of the oscillating and the contact zone models was made by Comninou (1990).

3. CONTACT ZONE LENGTH IN TERMS OF THE SIFs OF THE OSCILLATING MODEL

Equation (6) is valid only for the problem depicted in Fig. 1. Next we will find the equation for the contact zone length determination with respect to the SIFs of the oscillating model. Using formula (8) and introducing SIFs as Rice (1988)

$$K_1 + iK_2 = r^{-ie}\sqrt{2\pi r}(\sigma_y + i\sigma_{xy})|_{y=0, x\to b+0},$$

we get

$$K_1 = \sqrt{\frac{l\pi}{2}}[(\sigma - 2e\tau)\cos\varphi + (2e\sigma + \tau)\sin\varphi],$$

$$K_2 = -\sqrt{\frac{l\pi}{2}}[(\sigma - 2e\tau)\sin\varphi - (2e\sigma + \tau)\cos\varphi],$$

where $\varphi = e\ln(l)$.

From these equations the value of $\delta = \tau/\sigma$ can be expressed in the form

$$\delta = \frac{K_2/K_1 - 2e + (2eK_2/K_1 + 1)\tan\varphi}{2eK_2/K_1 + 1 - (K_2/K_1 - 2e)\tan\varphi}.$$

Substituting the last expression into the eqn (6) gives the following equation:

$$\beta = g + \tan^{-1}\frac{2e(1-\sqrt{1-\lambda})}{4e^2 + \sqrt{1-\lambda}},$$

$$g = -e\ln(l) - \psi + \pi(m - 0.5), \quad \psi = \tan^{-1}(K_2/K_1) \tag{12}$$

with respect to λ. Particularly for $\lambda \ll 1$ this equation approximately can be written as $\beta = g$ and the exact solution $\bar{\lambda}_0 \approx \lambda_0$ of this equation is

$$\bar{\lambda}_0 = 1 - \tanh^2(-g/2e), \quad (g < 0). \tag{13}$$

If we introduce the SIFs by the Erdogan and Gupta (1971) formula

$$K_1^* + iK_2^* = \left(\frac{r}{l}\right)^{-ie}\sqrt{2\pi r}(\sigma_y + i\sigma_{xy})|_{y=0, x\to b+0},$$

then

$$K_1^* = \sqrt{\frac{l\pi}{2}}(\sigma - 2e\tau), \quad K_2^* = \sqrt{\frac{l\pi}{2}}(2e\sigma + \tau)$$

and in this case

$$\delta = \frac{K_2^*/K_1^* - 2e}{2eK_2^*/K_1^* + 1}.$$

Substituting the last formula into the eqn (6) we get the following equation

$$\beta = g^* + \tan^{-1}\frac{2e(1-\sqrt{1-\lambda})}{4e^2+\sqrt{1-\lambda}}, \tag{14}$$

where $g^* = -\psi^* + \pi(m-0.5)$, $\psi^* = \tan^{-1}(K_2^*/K_1^*)$. For $\lambda \ll 1$ the last equation can be approximately written in the form $\beta = g^*$ and the exact solution of this equation is

$$\lambda_0^* = 1 - \tanh^2(-g^*/2e). \tag{15}$$

The value of m in the relations (12)–(15) should be taken properly to get the maximum value of the contact zone length.

The obtained eqns (12) and (14) are exact for the bimaterial plane and the formulas (13) and (15) are their asymptotic solutions. But these equations and formulas can be used as well for an interface crack in a finite size body, due to SIFs completely define the local field near a crack tip. That is why if the contact zone length λ_0 is small with respect to 1, which is true in many cases, then the value of λ_0 can be found with high accuracy by means of the eqns (12) and (14) or more roughly with the formulas (13) and (15). As for the relatively long contact zone ($\lambda_0 > 0.01$) then the mentioned equations and formulas can be used as an initial approximation for the contact zone length determination, if this length is far less than the characteristic dimension of the considered problem (size of the body, distance between the crack tip and the boundary of the region, etc.) It is clear of course that the SIFs K_1, K_2 in a finite size body can be found as a rule in a numerical manner.

Example 1: For an interface crack of the length l between the semi-infinite plane ($E_1 = 10^7$, $\nu_1 = 0.3$) and the strip ($E_2 = 4.5 \cdot 10^5$, $\nu_2 = 0.35$) of width h under an internal pressure Erdogan and Gupta (1971) obtained the SIFs K_1^*, K_2^* depicted in figure 6 of the mentioned work. On the base of these results we have K_2^*/K_1^* approximately equal to -0.134, 0.0, 0.175 for h/l equal to ∞, 0.45, 0.2, respectively, and by means of the formula (15) ($e < 0, m = 1$) we get the correspondent values of the relative contact zone lengths λ_0^* as $3.36 \cdot 10^{-11}$, $2.45 \cdot 10^{-10}$, $3.28 \cdot 10^{-9}$.

Example 2: Consider now Brasil-nut sandwich specimen composed of two circular half-discs (μ_1, ν_1) of radius R and thin interlayer (μ_2, ν_2) of the thickness H. An interface crack of the length l exists between the top of the interlayer and the upper half of the dick. The dick is loaded in compression along the diameter at an angle ϕ_0 to the crack plane with the two concentrated forces P. For $\nu_1 = \nu_2 = 0.3$, various μ_1/μ_2, ϕ_0, l/R and H/R the values of ψ^* for the left crack tip are found in the paper by Bois-Grossian and Tan (1995). Some of these results for $H/l = 0.1$, $l/R = 0.5$ are reproduced in the upper part of each line of Table 3. Besides in the lower part of each line the correspondent values of the relative contact zone length λ_0^*, obtained with the formula (15) are shown. The interlayer was

Table 3. The values of $\psi^* = \tan^{-1}(K_2^*/K_1^*)$ (the upper dates in each square) and the correspondent relative contact zone lengths λ_0^* (the lower dates)

	ϕ_0		
μ_1/μ_2	5°	10°	15°
2	-0.322	-0.666	-0.924
	$4.60 \cdot 10^{-18}$	$2.91 \cdot 10^{-13}$	$2.38 \cdot 10^{-9}$
5	-0.383	-0.682	-0.889
	$1.61 \cdot 10^{-8}$	$2.20 \cdot 10^{-6}$	$6.12 \cdot 10^{-5}$
10	-0.527	-0.750	-0.942
	$4.18 \cdot 10^{-6}$	$7.93 \cdot 10^{-5}$	$1.04 \cdot 10^{-3}$

considered as material "1", half-dick as material "2" and the mentioned crack tip becomes the right crack tip, and the value of m in the formula (15) was taken to be equal to 0. In a similar way a relative contact zone length for any ψ^* can be found.

4. THE ENERGY RELEASE RATES FOR THE CONTACT ZONE AND OSCILLATORY MODELS

For the crack shown in Fig. 1 the ERR can be computed as the virtual work integral

$$G = \lim_{\Delta l \to 0}\left[\frac{1}{2\Delta l}\int_a^{a-\Delta l}\bar{\sigma}_y(x)\bar{v}(x+\Delta l)\,\mathrm{d}x+\frac{1}{2\Delta l}\int_b^{b+\Delta l}\bar{\sigma}_{xy}(x)\bar{u}(x+\Delta l)\,\mathrm{d}x\right], \tag{16}$$

where

$$\bar{\sigma}_y(x) = \sigma_y(x,0) \quad \text{for } x \to a+0,$$
$$\bar{v}(x) = v(x,0) \quad \text{for } x \to a-0,$$
$$\bar{\sigma}_{xy}(x) = \sigma_{xy}(x,0) \quad \text{for } x \to b+0,$$
$$\bar{u}(x) = u(x,0) \quad \text{for } x \to b-0.$$

Using the asymptotic expressions for stress and displacement derivative fields near singular points $z = a+i0$ and $z = b+i0$ obtained by Loboda (1993) the following formulas have been found

$$\bar{\sigma}_y(x) = \frac{k_1}{\sqrt{2(x-a)}}, \quad \bar{v}(x) = \frac{\kappa_1+1}{4\mu_1}\sqrt{2(a-x)}k_1,$$
$$\bar{\sigma}_{xy}(x) = \frac{k_2}{\sqrt{2(x-b)}}, \quad \bar{u}(x) = \frac{(\kappa_1+\gamma)\sqrt{2(b-x)}}{2\mu_1(1+\gamma)}k_2.$$

Substituting of the last formulas into (16) and evaluation of the integrals leads to the following result

$$G(\lambda) = \frac{\pi q}{4}(\alpha k_1^2+k_2^2), \tag{17}$$

where

$$\alpha = (\gamma+1)^2/(4\gamma), \quad q = \frac{(\mu_1+\mu_2\kappa_1)(\mu_2+\mu_1\kappa_2)}{\mu_1\mu_2(\mu_1+\mu_2+\mu_2\kappa_1+\mu_1\kappa_2)},$$

which is the same as $J(\lambda)$ in the just mentioned paper.

By using of the expression (4) for the SIFs k_1 and k_2 the formula (17) can be presented in the form

$$G(\lambda) = \frac{\pi q}{8}(1+4e^2)l[\sigma^2+\tau^2-\lambda(\sigma\cos\beta-\tau\sin\beta)^2]. \tag{18}$$

Taking into account that for $\lambda = \lambda_0$

$$\tan\beta_0 = \frac{r_0+2e\delta}{2e-\delta r_0}, \quad r_0 = \sqrt{1-\lambda_0} \tag{19}$$

after trigonometric simplifications we arrive

$$G(\lambda_0) = \frac{\pi q}{4} k_{20}^2 = \frac{\pi q}{8}(1+4e^2) l(\sigma^2+\tau^2)\left(1-\frac{4e^2\lambda_0}{1+4e^2-\lambda_0}\right) \tag{20}$$

where k_{20} is the value of k_2 for $\lambda = \lambda_0$. It is worth noting that the values of k_{20} were previously obtained in numerical manner by Comninou and Schmueser (1979) and analytically but in more complicated form by Gautesen and Dundurs (1988a, b). Comparison of

$$k_{20}/(\tau\sqrt{0.5l}) = \sqrt{(1+4e^2)\left(1+\frac{\sigma^2}{\tau^2}\right)\left(1-\frac{4e^2\lambda_0}{1+4e^2-\lambda_0}\right)} \tag{21}$$

with the mentioned results showed that for $\gamma = 3$ the values of Gautesen and Dundurs (1988a) are 1.032, 1.071 and 1.138 for $\sigma/\tau = 0$, -0.2, -0.4, respectively, whereas formula (21) gives 1.0317, 1.0703 and 1.1369. The expression (20) is the ERR for the contact zone model in terms of a remote traction-shear field.

Now we consider the ERR for the oscillating model. Using for this purpose formula

$$G_{0s} = \frac{Q_1^2+Q_2^2}{4\cosh^2(\pi e)}\left[\frac{1-\nu_1}{\mu_1}+\frac{1-\nu_2}{\mu_2}\right]$$

from the paper by Shih and Asaro (1988) and taking into account that

$$Q_1+iQ_2 = [(\sigma-2\tau e)+i(\tau+2\sigma e)]\sqrt{\pi l/2},$$

we obtain

$$G_{0s} = \frac{\pi q}{8}(1+4e^2) l(\sigma^2+\tau^2). \tag{22}$$

It is clear that G_{0s} could be found by assuming $\lambda_0 = 0$ in (20), i.e. $G_{0s} = G(0)$.

It is worth comparing the values of ERR obtained by using two models. Relative difference between these values can be determined by

$$\delta G = \frac{G_{0s}-G(\lambda_0)}{G_{0s}} = \frac{4e^2\lambda_0}{1+4e^2-\lambda_0}. \tag{23}$$

For the only tension field ($\tau = 0, \lambda_0 = O(10^{-4})$) the value $|\delta G|$ is negligible small of order 10^{-6} for any material combinations. But it is interesting that this difference is rather small even for an essential contact zone length. For example for the extreme situation when $\delta \to -\infty$ and $\gamma = 3$ we have: $\lambda_0 \approx 0.3291$, $e \approx 0.1748$ and consequently $\delta G \approx 0.0507$. It means that even in this case the difference in the ERR determination by using the two models is not very essential. Since for practically real material combinations the values $e \leqslant 0.1$ (Rice, 1988), δG is extremely small for any δ and both the contact zone model and the oscillating model can be used for the ERR determination.

5. QUASI-INVARIANCE OF THE ERR WITH RESPECT TO THE CONTACT ZONE LENGTH

The oscillating model in spite of its simplicity is not explicitly convenient for the numerical fracture parameters determination due to complex behavior of stresses and displacements at the crack tip. On the other hand the contact model utilization in its conventional form requires λ_0 definition and only after that the value of K_{20} or $G(\lambda_0)$ determination. This way leads to the complex nonlinear problem which cannot be easily solved for a finite size body containing an interface crack. Essential simplification of this procedure can be obtained by using quasi-invariance of ERR $G(\lambda)$ with respect to λ investigated for a small λ_0 by Loboda (1993). Now we consider an arbitrary value of λ_0. According to (18)

$$\delta G(\lambda) = \frac{G(\lambda)-G(\lambda_0)}{G(\lambda_0)} = \frac{\lambda_0(\sigma\cos\beta_0-\tau\sin\beta_0)^2-\lambda(\sigma\cos\beta-\tau\sin\beta)^2}{\sigma^2+\tau^2-\lambda_0(\sigma\cos\beta_0-\tau\sin\beta_0)^2}. \tag{24}$$

Using Taylor's series for $(\sigma\cos\beta-\tau\sin\beta)^2$ at the point λ_0 and taking into account that due to (19)

$$\sigma\cos\beta_0-\tau\sin\beta_0 = \frac{2e\sigma\sqrt{1+\delta^2}}{\sqrt{1-\lambda_0+4e^2}},$$

$$\sigma\sin\beta_0+\tau\cos\beta_0 = \frac{\sigma\sqrt{1-\lambda_0}\sqrt{1+\delta^2}}{\sqrt{1-\lambda_0+4e^2}},$$

we arrive

$$\delta G(\lambda) = \frac{1-2\lambda_0+4e^2}{\lambda_0(1-\lambda_0)^2(1+4e^2)}e^2(\lambda-\lambda_0)^2+O|\lambda-\lambda_0|^3 \tag{25}$$

[we notice that the coefficient at the $(\lambda-\lambda_0)$ is equal to zero].

The last formula can be applied both for $\lambda_0<\lambda$ and $\lambda_0>\lambda$. But for a small λ_0 and $\lambda_0<\lambda\ll 1$ formula (25) is not convenient. It is this case directly from (24) due to obvious inequality $(\sigma\cos\beta-\tau\sin\beta)^2\leqslant\sigma^2+\tau^2$ we obtain

$$0\leqslant -\delta G(\lambda)\leqslant\frac{\lambda}{1-\lambda_0}-\lambda_0\frac{\lambda+4e^2}{(1+4e^2)(1-\lambda_0)}\leqslant\frac{\lambda}{1-\lambda_0}. \tag{26}$$

It follows from (25) and (26) that

$$|\delta G(\lambda)| = \begin{cases} O(\lambda), & \text{for a small } \lambda_0<\lambda\ll 1, \\ O[(\lambda-\lambda_0)^2], & \text{for the remaining } \lambda_0. \end{cases} \tag{27}$$

The formula (27) declares the quasi-invariance of $G(\lambda)$ in some vicinity $|\lambda-\lambda_0|<\varepsilon$ of λ_0. This quality eliminates necessity in the precise λ_0 definition for $G(\lambda_0)$ determination, but permit to find $G(\lambda)$ for any λ from ε-vicinity of λ_0. After assuming $G(\lambda_0)\approx G(\lambda)$ we will make an error (mistake) of order ε^2 (ε for a small λ_0). For the most practically important weak shear field (small λ_0) we may directly take $\lambda=0.01$. In this case the error in $G(\lambda_0)$ determination by means of $G(\lambda)$ calculation will not exceed 1%.

For an essential value of $\lambda_0\geqslant 0.02$ the needed ε-vicinity can be found by means of an iterative solution of a problem in question for $\lambda\neq\lambda_0$ with control of the value K_1.

6. CONCLUSION

On the base of the exact analytical solution for a crack between two semi-infinite planes with the frictionless contact zone of arbitrary length two interface crack models—the oscillating one and the contact zone one are derived. The procedure of transition from one model to another is demonstrated and approximate relationship between the contact zone lengths is obtained.

The transendental equation and the asymptotic formulas for the contact zone length determination with respect to the SIFs of the oscillating model are found. These relations give possibility to find contact zone length not only via remote tension–shear stress components but via the SIFs which can be usually found for the various problems in a numerical manner.

The simple formula for the ERR of the contact model is found and compared with the ERR of the oscillating model. A small difference between the mentioned values is obtained even for the essential contact zone length.

The quasi-invariance of the ERR for a crack with a contact zone of arbitrary relative length λ with respect to λ is proved in the analytical manner. On the base of this phenomenon the easier way of the contact zone model application to the finite size composites investigation is suggested. With a slight modification this way can be used both for a short and long contact zone lengths.

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